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Multipartite entangled state representation and squeezing of the *n*-pair entangled state

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Abstract
We show that the best way to study the relationship between quantum entanglement and quantum squeezing for the multimode case is through constructing the new n-pair entangled state representation $|\boldsymbol{\eta}\rangle_n$. The explicit form of the $2n$-mode squeezing operator which squeezes the n-pair entangled state can be directly derived via the transition from $|\boldsymbol{\eta}\rangle_n$ to $|\mathbf{M}\boldsymbol{\eta}\rangle_n$, where $\mathbf{M}$ is an $n \times n$ complex matrix, and the technique of integration within an ordered product of operators. We also analyze the squeezing properties of the $2n$-mode squeezed state for $\mathbf{M}$ being Hermitian, and obtain the variances of the $2n$-mode quadrature operators in the $2n$-mode squeezed vacuum state with a concise result; the condition for reaching the minimum of the uncertainty relationship is also investigated.

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1. Introduction

Recently, quantum entanglement has been paid much attention because of its applications in quantum optics [1, 2] and quantum information. It was first pointed out by Einstein, Podolsky and Rosen (EPR) [3] in their famous paper arguing the incompleteness of quantum mechanics. EPR introduced the common eigenwavefunction for two particles' relative position $Q_1 - Q_2$ (with the relative distance Q_0) and their total momentum $P_1 + P_2$ (with the eigenvalue P_0),

$$\Psi(Q_1, Q_2) = \exp[\mathrm{i}P_0(Q_1 + Q_2)/2]\frac{1}{2\pi}\int \mathrm{d}p\,\exp[\mathrm{i}p(Q_1 - Q_2 - Q_0)], \quad (1)$$

which describes a sharply correlated two-particle system. Enlightened by EPR, in [4, 5] the simultaneous eigenstate $|\eta\rangle$ of the two commutative operators $Q_1 - Q_2$, $P_1 + P_2$ is found in two-mode Fock space:

$$|\eta\rangle = \exp\left[-\tfrac{1}{2}|\eta|^2 + \eta a_1^\dagger - \eta^* a_2^\dagger + a_1^\dagger a_2^\dagger\right]|00\rangle, \quad (2)$$

where $\eta = \eta_1 + \mathrm{i}\eta_2$ is a complex number, $|00\rangle$ is the two-mode vacuum state, $a_i, a_i^\dagger, i = 1, 2$, are the two-mode Bose annihilation and creation operators in Fock space, $|00\rangle\langle 00| =: \exp\left[-a_1^\dagger a_1 - a_2^\dagger a_2\right]:$. With the help of the technique of integration within an ordered product (IWOP) of operators [6], we have the completeness of the states $|\eta\rangle$:

$$\begin{aligned}\int \frac{\mathrm{d}^2\eta}{\pi}|\eta\rangle\langle\eta| &= \int \frac{\mathrm{d}^2\eta}{\pi} \exp\left[-|\eta|^2 + \eta a_1^\dagger - \eta^* a_2^\dagger + a_1^\dagger a_2^\dagger\right]|00\rangle\langle 00| \exp[\eta^* a_1 - \eta^* a_2 + a_1 a_2] \\ &= \int \frac{\mathrm{d}^2\eta}{\pi} : \exp\left[-|\eta|^2 + \eta\left(a_1^\dagger - a_2\right) - \eta^*\left(a_2^\dagger + a_1\right) + a_1^\dagger a_2^\dagger + a_1 a_2 - a_1^\dagger a_1 - a_2^\dagger a_2\right]: \\ &= 1. \end{aligned} \tag{3}$$

Furthermore, the states $|\eta\rangle$ are orthonormal, i.e.

$$\langle\eta'|\eta\rangle = \pi\delta^{(2)}(\eta' - \eta). \tag{4}$$

It is remarkable that the two-mode squeezing operator [9, 13] has a natural representation in the entangled state $|\eta\rangle$ [7]:

$$S_2 \equiv \int \frac{\mathrm{d}^2\eta}{\mu\pi}\left|\frac{\eta}{\mu}\right\rangle\langle\eta|, \tag{5}$$

where $\mu = \mathrm{e}^\lambda$ is a real squeezing parameter. This fact provides an intuitive explanation as to why the signal mode and the idler mode of a two-mode squeezed state produced from a parametric down-conversion process are entangled with each other and constitute an entangled state.

Using the technique of IWOP to perform the integration in equation (5) we have

$$S_2 = \exp\left[\lambda\left(a_1^\dagger a_2^\dagger - a_1 a_2\right)\right], \qquad S_2|\eta\rangle = \frac{1}{\mu}\left|\frac{\eta}{\mu}\right\rangle, \tag{6}$$

so the two-mode squeezing operator squeezes $|\eta\rangle$ in a natural way. This indicates that at least for the two-mode case squeezing is associated with quantum entanglement. An interesting question thus naturally arises: is there a $2n$-mode squeezing operator which squeezes the n-pair entangled state? If yes, what is the corresponding explicit form of the $2n$-mode squeezing operator?

In order to answer these questions, hinted by equation (6), we should employ the technique of IWOP to construct the n-pair entangled state; this would bring convenience for studying the $2n$-mode squeezing. This paper is arranged as follows. In section 2, we construct the n-pair entangled state representation $|\boldsymbol{\eta}\rangle_n$. In section 3, we introduce the $2n$-mode squeezing operator in the $|\boldsymbol{\eta}\rangle_n$ representation by constructing the bra-ket integration operator $U(\mathbf{M}) = \sqrt{\det(\mathbf{M}^\dagger\mathbf{M})}\int \frac{\mathrm{d}^2\eta}{\pi^n}|\mathbf{M}\boldsymbol{\eta}\rangle_{nn}\langle\boldsymbol{\eta}|$, where $\mathbf{M}$ is an $n \times n$ nonsingular complex matrix, and then use the IWOP technique to derive its normally ordered form. Then in section 4, the explicit form of the $2n$-mode squeezing operator in the $|\boldsymbol{\xi}\rangle_n$ representation, which is conjugate to $|\boldsymbol{\eta}\rangle_n$, is also presented. In order to show the squeezing behavior more clearly, in section 5 we present the compact form of the $2n$-mode squeezing operator and analyze the squeezing properties for $\mathbf{M}$ being Hermitian. In section 6, we derive the variances of the $2n$-mode quadrature operators in a $2n$-mode squeezed vacuum state with a concise result; the condition for reaching the minimum of the uncertainty relationship is also investigated.

2. n-pair entangled state representation $|\eta\rangle_n$ in Fock space

First of all, we introduce the n-pair entangled state

$$|\boldsymbol{\eta}\rangle_n \equiv |\eta_1\rangle|\eta_2\rangle\cdots|\eta_n\rangle \equiv \left|\begin{pmatrix}\eta_1\\ \eta_2\\ \vdots\\ \eta_n\end{pmatrix}\right\rangle = \exp\left(-\frac{1}{2}\boldsymbol{\eta}^{*T}\boldsymbol{\eta} + \mathbf{A}^{\dagger T}\boldsymbol{\eta} - \mathbf{B}^{\dagger T}\boldsymbol{\eta}^* + \mathbf{A}^{\dagger T}\mathbf{B}^{\dagger}\right)|\mathbf{00}\rangle, \tag{7}$$

where

$$\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_n)^T, \qquad \boldsymbol{\eta}^* = (\eta_1^*, \eta_2^*, \ldots, \eta_n^*)^T,$$

$\boldsymbol{\eta}_k = \eta_{1k} + \mathrm{i}\eta_{2k}, k = 1, 2, \ldots, n,$ are complex numbers,

$$\mathbf{A}^{\dagger} = \left(a_1^{\dagger}, a_3^{\dagger}, \ldots, a_{2n-1}^{\dagger}\right)^T, \qquad \mathbf{B}^{\dagger} = \left(a_2^{\dagger}, a_4^{\dagger}, \ldots, a_{2n}^{\dagger}\right)^T,$$

and $|\mathbf{00}\rangle$ is a $2n$-mode vacuum state, $|\mathbf{00}\rangle\langle\mathbf{00}| =: \exp[-\mathbf{A}^{\dagger T}\mathbf{A} - \mathbf{B}^{\dagger T}\mathbf{B}]:$. Here and henceforth, the superscript 'T' indicates the transpose matrix of the corresponding matrix. By virtue of equation (3), the completeness of the states $|\boldsymbol{\eta}\rangle_n$ can be proved, i.e.

$$\begin{aligned}\int \frac{\mathrm{d}^2\boldsymbol{\eta}}{\pi^n}|\boldsymbol{\eta}\rangle_{nn}\langle\boldsymbol{\eta}| &= \int \frac{\mathrm{d}^2\boldsymbol{\eta}}{\pi^n}\exp(-\boldsymbol{\eta}^{*T}\boldsymbol{\eta} + \mathbf{A}^{\dagger T}\boldsymbol{\eta} - \mathbf{B}^{\dagger T}\boldsymbol{\eta}^*\\ &\quad + \mathbf{A}^{\dagger T}\mathbf{B}^{\dagger})|\mathbf{00}\rangle\langle\mathbf{00}|\exp(\mathbf{A}^T\boldsymbol{\eta}^* - \mathbf{B}^T\boldsymbol{\eta} + \mathbf{A}^T\mathbf{B})\\ &= \int\frac{\mathrm{d}^2\boldsymbol{\eta}}{\pi^n} : \exp[-\boldsymbol{\eta}^{*T}\boldsymbol{\eta} + (\mathbf{A}^{\dagger T} - \mathbf{B}^T)\boldsymbol{\eta}\\ &\quad + (\mathbf{A}^T - \mathbf{B}^{\dagger T})\boldsymbol{\eta}^* + \mathbf{A}^{\dagger T}\mathbf{B}^{\dagger} + \mathbf{A}^T\mathbf{B} - \mathbf{A}^{\dagger T}\mathbf{A} - \mathbf{B}^{\dagger T}\mathbf{B}]:\\ &= 1;\end{aligned} \tag{8}$$

here and henceforth we define $\mathrm{d}^2\boldsymbol{\eta} = \mathrm{d}^2\eta_1\,\mathrm{d}^2\eta_2\cdots\mathrm{d}^2\eta_n$ for simplicity. Using equation (4), we demonstrate that $|\boldsymbol{\eta}\rangle_n$ are orthonormal, i.e.

$${}_n\langle\boldsymbol{\eta}'|\boldsymbol{\eta}\rangle_n = \pi^n\delta^{(2)}(\eta_1' - \eta_1)\delta^{(2)}(\eta_2' - \eta_2)\cdots\delta^{(2)}(\eta_n' - \eta_n) = \pi^n\delta(\boldsymbol{\eta}' - \boldsymbol{\eta}). \tag{9}$$

The eigenequations satisfied by the state $|\boldsymbol{\eta}\rangle_n$ are

$$(\mathbf{A} - \mathbf{B}^{\dagger})|\boldsymbol{\eta}\rangle_n = \boldsymbol{\eta}|\boldsymbol{\eta}\rangle_n, \qquad (\mathbf{B} - \mathbf{A}^{\dagger})|\boldsymbol{\eta}\rangle_n = -\boldsymbol{\eta}^*|\boldsymbol{\eta}\rangle_n. \tag{10}$$

Since

$$Q_k = \frac{1}{\sqrt{2}}\left(a_k + a_k^{\dagger}\right), \qquad P_k = \frac{1}{\sqrt{2}i}\left(a_k - a_k^{\dagger}\right), \tag{11}$$

we obtain

$$(Q_A - Q_B)|\boldsymbol{\eta}\rangle_n = \sqrt{2}\boldsymbol{\eta}_1|\boldsymbol{\eta}\rangle_n, \tag{12}$$

and

$$(P_A + P_B)|\boldsymbol{\eta}\rangle_n = \sqrt{2}\boldsymbol{\eta}_2|\boldsymbol{\eta}\rangle_n, \tag{13}$$

where $Q_A = (Q_1, Q_3, \ldots, Q_{2n-1})^T$, $Q_B = (Q_2, Q_4, \ldots, Q_{2n})^T$.

3. Normally ordered 2n-mode squeezing operator derived through the $|\eta\rangle_n$ representation

Enlightened by equation (5), we construct the following ket-bra integration operator which maps $|\boldsymbol{\eta}\rangle_n$ to $\sqrt{\det(\mathbf{M}^\dagger\mathbf{M})}|\mathbf{M}\boldsymbol{\eta}\rangle_n$ in the $|\boldsymbol{\eta}\rangle_n$ representation:

$$U(\mathbf{M}) = \sqrt{\det(\mathbf{M}^\dagger\mathbf{M})}\int\frac{\mathrm{d}^2\boldsymbol{\eta}}{\pi^n}|\mathbf{M}\boldsymbol{\eta}\rangle_{nn}\langle\boldsymbol{\eta}|, \tag{14}$$

where $\mathbf{M} = \mathbf{M}_1 + \mathrm{i}\mathbf{M}_2$ is an $n \times n$ nonsingular complex matrix. As usual, the Hermitian conjugate of $\mathbf{M}$ is defined as $\mathbf{M}^\dagger = \mathbf{M}_1^T - \mathrm{i}\mathbf{M}_2^T$. It is easy to see that $U(\mathbf{M})$ is unitary, i.e.

$$\begin{aligned}U^\dagger(\mathbf{M})U(\mathbf{M}) &= \det(\mathbf{M}^\dagger\mathbf{M})\int\frac{\mathrm{d}^2\boldsymbol{\eta}'\,\mathrm{d}^2\boldsymbol{\eta}}{\pi^{2n}}|\boldsymbol{\eta}'\rangle_{nn}\langle\mathbf{M}\boldsymbol{\eta}'|\mathbf{M}\boldsymbol{\eta}\rangle_{nn}\langle\boldsymbol{\eta}|\\ &= \det(\mathbf{M}^\dagger\mathbf{M})\int\frac{\mathrm{d}^2\boldsymbol{\eta}'\,\mathrm{d}^2\boldsymbol{\eta}}{\pi^{2n}}|\boldsymbol{\eta}'\rangle_{nn}\langle\boldsymbol{\eta}|\delta(\mathbf{M}(\boldsymbol{\eta}'-\boldsymbol{\eta}))\\ &= \int\frac{\mathrm{d}^2\boldsymbol{\eta}'\,\mathrm{d}^2\boldsymbol{\eta}}{\pi^{n}}|\eta'\rangle_{nn}\langle\eta|\delta(\boldsymbol{\eta}-\boldsymbol{\eta}')\\ &= 1.\end{aligned} \tag{15}$$

We also have the multiplication rule (group property, isomorphism)

$$\begin{aligned}U(\mathbf{M}')U(\mathbf{M}) &= \sqrt{\det(\mathbf{M}'^\dagger\mathbf{M}')}\sqrt{\det(\mathbf{M}^\dagger\mathbf{M})}\int\frac{\mathrm{d}^2\boldsymbol{\eta}'\,\mathrm{d}^2\boldsymbol{\eta}}{\pi^{2n}}|\mathbf{M}'\boldsymbol{\eta}'\rangle_{nn}\langle\boldsymbol{\eta}'|\mathbf{M}\boldsymbol{\eta}\rangle_{nn}\langle\boldsymbol{\eta}|\\ &= \sqrt{\det((\mathbf{M}'\mathbf{M})^\dagger\mathbf{M}'\mathbf{M})}\int\frac{\mathrm{d}^2\boldsymbol{\eta}'\,\mathrm{d}^2\boldsymbol{\eta}}{\pi^{n}}|\mathbf{M}'\boldsymbol{\eta}'\rangle_{nn}\langle\boldsymbol{\eta}|\delta(\boldsymbol{\eta}'-\mathbf{M}\boldsymbol{\eta})\\ &= \sqrt{\det((\mathbf{M}'\mathbf{M})^\dagger\mathbf{M}'\mathbf{M})}\int\frac{\mathrm{d}^2\boldsymbol{\eta}}{\pi^{n}}|\mathbf{M}'\mathbf{M}\boldsymbol{\eta}\rangle_{nn}\langle\boldsymbol{\eta}|\\ &= U(\mathbf{M}'\mathbf{M}).\end{aligned} \tag{16}$$

The transformation on $Q_A - Q_B$ and $P_A + P_B$ induced by $U(\mathbf{M})$ is

$$\begin{aligned}U^\dagger(\mathbf{M})(Q_A-Q_B)U(\mathbf{M}) &= U^\dagger(\mathbf{M})\sqrt{\det(\mathbf{M}^\dagger\mathbf{M})}\int\frac{\mathrm{d}^2\boldsymbol{\eta}}{\pi^n}(Q_A-Q_B)|\mathbf{M}\boldsymbol{\eta}\rangle_{nn}\langle\boldsymbol{\eta}|\\ &= U^\dagger(\mathbf{M})\sqrt{\det(\mathbf{M}^\dagger\mathbf{M})}\int\frac{\mathrm{d}^2\boldsymbol{\eta}}{\pi^n}\sqrt{2}(\mathbf{M}\boldsymbol{\eta})_1|\mathbf{M}\boldsymbol{\eta}\rangle_{nn}\langle\boldsymbol{\eta}|\\ &= U^\dagger(\mathbf{M})\sqrt{\det(\mathbf{M}^\dagger\mathbf{M})}\int\frac{\mathrm{d}^2\boldsymbol{\eta}}{\pi^n}|\mathbf{M}\boldsymbol{\eta}\rangle_{nn}\langle\boldsymbol{\eta}|\sqrt{2}\,(\mathbf{M}_1\boldsymbol{\eta}_1-\mathbf{M}_2\boldsymbol{\eta}_2)\\ &= \mathbf{M}_1(Q_A-Q_B)-\mathbf{M}_2(P_A+P_B)\end{aligned} \tag{17}$$

and

$$\begin{aligned}U^\dagger(\mathbf{M})(P_A+P_B)U(\mathbf{M}) &= U^\dagger(\mathbf{M})\sqrt{\det(\mathbf{M}^\dagger\mathbf{M})}\int\frac{\mathrm{d}^2\boldsymbol{\eta}}{\pi^n}(P_A+P_B)|\mathbf{M}\boldsymbol{\eta}\rangle_{nn}\langle\boldsymbol{\eta}|\\ &= U^\dagger(\mathbf{M})\sqrt{\det(\mathbf{M}^\dagger\mathbf{M})}\int\frac{\mathrm{d}^2\boldsymbol{\eta}}{\pi^n}\sqrt{2}(\mathbf{M}\boldsymbol{\eta})_2|\mathbf{M}\boldsymbol{\eta}\rangle_{nn}\langle\boldsymbol{\eta}|\\ &= U^\dagger(\mathbf{M})\sqrt{\det(\mathbf{M}^\dagger\mathbf{M})}\int\frac{\mathrm{d}^2\boldsymbol{\eta}}{\pi^n}\sqrt{2}|\mathbf{M}\boldsymbol{\eta}\rangle_{nn}\langle\boldsymbol{\eta}|\,(\mathbf{M}_1\boldsymbol{\eta}_2+\mathbf{M}_2\boldsymbol{\eta}_1)\\ &= \mathbf{M}_1(P_A+P_B)+\mathbf{M}_2(Q_A-Q_B).\end{aligned} \tag{18}$$

We can combine equations (17) and (18) as a more compact expression

$$U^\dagger(\mathbf{M})\mathbf{O}U(\mathbf{M}) = \mathbf{M}\mathbf{O}, \tag{19}$$

where

$$\mathbf{O} = (Q_A - Q_B) + \mathrm{i}(P_A + P_B). \tag{20}$$

That is, if we pretend that $Q_A - Q_B$ and $P_A + P_B$ are real, and separate the 'real' and 'imaginary' parts of both sides of equation (19), we retrieve equations (17) and (18).

To get the explicit form of $U(\mathbf{M})$, we use the IWOP technique

$$\begin{aligned} U(\mathbf{M}) &= \sqrt{\det(\mathbf{M}^\dagger\mathbf{M})}\int \frac{\mathrm{d}^2\boldsymbol{\eta}}{\pi^n}|\mathbf{M}\boldsymbol{\eta}\rangle_{nn}\langle\boldsymbol{\eta}| \\ &= \sqrt{\det(\mathbf{M}^\dagger\mathbf{M})}\int \frac{\mathrm{d}^2\boldsymbol{\eta}}{\pi^n}\exp\left(-\frac{1}{2}\boldsymbol{\eta}^{*T}\mathbf{M}^\dagger\mathbf{M}\boldsymbol{\eta} + \mathbf{A}^{\dagger T}\mathbf{M}\boldsymbol{\eta} - \mathbf{B}^{\dagger T}\mathbf{M}^*\boldsymbol{\eta}^* + \mathbf{A}^{\dagger T}\mathbf{B}^\dagger\right)|\mathbf{00}\rangle\langle\mathbf{00}| \\ &\quad\times\exp\left(-\frac{1}{2}\boldsymbol{\eta}^{*T}\boldsymbol{\eta} + \mathbf{A}^T\boldsymbol{\eta}^* - \mathbf{B}^T\boldsymbol{\eta} + \mathbf{A}^T\mathbf{B}\right) \\ &= \sqrt{\det(\mathbf{M}^\dagger\mathbf{M})}\int \frac{\mathrm{d}^2\boldsymbol{\eta}}{\pi^n} \\ &\quad :\exp\begin{bmatrix} -\frac{1}{2}\boldsymbol{\eta}^{*T}(\mathbf{I}+\mathbf{M}^\dagger\mathbf{M})\boldsymbol{\eta} + (\mathbf{A}^{\dagger T}\mathbf{M} - \mathbf{B}^T)\boldsymbol{\eta} - (\mathbf{B}^{\dagger T}\mathbf{M}^* - \mathbf{A}^T)\boldsymbol{\eta}^* \\ +\mathbf{A}^{\dagger T}\mathbf{B}^\dagger + \mathbf{A}^T\mathbf{B} - \mathbf{A}^{\dagger T}\mathbf{A} - \mathbf{B}^{\dagger T}\mathbf{B}, \end{bmatrix}:, \end{aligned} \tag{21}$$

where $\mathbf{I}$ is the $n\times n$ unit matrix. Further, using the integral formula

$$\int \frac{\mathrm{d}^{2n}\mathbf{Z}}{\pi^n}\exp(-\mathbf{Z}^{*T}\zeta Z + \boldsymbol{\xi}^T\mathbf{Z} + \eta^T\mathbf{Z}^*) = \frac{1}{\det\zeta}\exp[\boldsymbol{\xi}^T\zeta^{-1}\eta],$$

where $\mathrm{d}^{2n}\mathbf{Z} = \mathrm{d}z_{11}\,\mathrm{d}z_{12}\,\mathrm{d}z_{21}\,\mathrm{d}z_{22}\cdots\mathrm{d}z_{1n}\,\mathrm{d}z_{2n}$, $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_n)^T$, $\boldsymbol{\zeta}$ is an $n\times n$ positive-definite matrix, $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are the complex column matrices, we perform the final integral in equation (21) and obtain the explicit form of $U(\mathbf{M})$

$$\begin{aligned} U(\mathbf{M}) &= \frac{2^n\sqrt{\det(\mathbf{M}^\dagger\mathbf{M})}}{\det(\mathbf{1}+\mathbf{M}^\dagger\mathbf{M})} :\exp[-2(\mathbf{A}^{\dagger T}\mathbf{M} - \mathbf{B}^T)(\mathbf{1}+\mathbf{M}^\dagger\mathbf{M})^{-1}(\mathbf{M}^\dagger\mathbf{B}^\dagger - \mathbf{A}) \\ &\quad + \mathbf{A}^{\dagger T}\mathbf{B}^\dagger + \mathbf{A}^T\mathbf{B} - \mathbf{A}^{\dagger T}\mathbf{A} - \mathbf{B}^{\dagger T}\mathbf{B}]: \\ &= \frac{2^n\sqrt{\det(\mathbf{M}^\dagger\mathbf{M})}}{\det(\mathbf{1}+\mathbf{M}^\dagger\mathbf{M})}\exp[\mathbf{A}^{\dagger T}(1 - 2\mathbf{M}(\mathbf{1}+\mathbf{M}^\dagger\mathbf{M})^{-1}\mathbf{M}^\dagger)\mathbf{B}^\dagger] \\ &\quad\cdot\exp[\mathbf{A}^{\dagger T}\ln(2\mathbf{M}(\mathbf{1}+\mathbf{M}^\dagger\mathbf{M})^{-1})\mathbf{A} + \mathbf{B}^{\dagger T}\ln(2\mathbf{M}^*(\mathbf{1}+\mathbf{M}^T\mathbf{M}^*)^{-1})\mathbf{B}] \\ &\quad\cdot\exp\left[\mathbf{B}^T\frac{\mathbf{M}^\dagger\mathbf{M} - \mathbf{1}}{\mathbf{1}+\mathbf{M}^\dagger\mathbf{M}}\mathbf{A}\right], \end{aligned} \tag{22}$$

where we have applied the operator identity

$$\exp[\mathbf{A}^{\dagger T}K\mathbf{A}] =: \exp[\mathbf{A}^{\dagger T}(\exp K - 1)\mathbf{A}]:.$$

4. The form of 2*n*-mode squeezing operator $U(\mathbf{M})$ in the $|\xi\rangle_n$ representation

We construct another 2*n*-mode entangled state as

$$|\boldsymbol{\xi}\rangle_n \equiv |\xi_1\rangle\,|\xi_2\rangle\cdots|\xi_n\rangle \equiv \left|\begin{pmatrix}\xi_1\\ \xi_2\\ \vdots\\ \xi_n\end{pmatrix}\right\rangle = \exp\left(-\frac{\boldsymbol{\xi}^{*T}\boldsymbol{\xi}}{2} + \mathbf{A}^{\dagger T}\boldsymbol{\xi} + \mathbf{B}^{\dagger T}\boldsymbol{\xi}^* - \mathbf{A}^{\dagger T}\mathbf{B}^\dagger\right)|\mathbf{00}\rangle, \tag{23}$$

where

$$\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)^T, \qquad \boldsymbol{\xi}^* = (\xi_1^*, \xi_2^*, \ldots, \xi_n^*)^T,$$

$\xi_k = \xi_{1k} + \mathrm{i}\xi_{2k}, k = 1, 2, \ldots, n,$

$$\mathbf{A}^\dagger = \left(a_1^\dagger, a_3^\dagger, \ldots, a_{2n-1}^\dagger\right)^T, \qquad \mathbf{B}^\dagger = \left(a_2^\dagger, a_4^\dagger, \ldots, a_{2n}^\dagger\right)^T.$$

Similarly, $|\boldsymbol{\xi}\rangle_n$ are complete and orthonormal:

$$\int \frac{\mathrm{d}^2\boldsymbol{\xi}}{\pi^n}|\boldsymbol{\xi}\rangle_{nn}\langle\boldsymbol{\xi}| = 1 \\ {}_n\langle\boldsymbol{\xi}'|\boldsymbol{\xi}\rangle_n = \pi^n\delta(\boldsymbol{\xi}' - \boldsymbol{\xi}). \tag{24}$$

The relationship between $|\boldsymbol{\xi}\rangle_n$ and $|\boldsymbol{\eta}\rangle_n$ is

$$ {}_n\langle\boldsymbol{\xi}|\boldsymbol{\eta}\rangle_n = \frac{1}{2^n}\exp\left[-\mathrm{i}\left(\boldsymbol{\eta}_1^T\boldsymbol{\xi}_2 - \boldsymbol{\eta}_2^T\boldsymbol{\xi}_1\right)\right], \\ |\boldsymbol{\eta}\rangle_n = \int\frac{\mathrm{d}^2\boldsymbol{\xi}}{\pi^n}|\boldsymbol{\xi}\rangle_{nn}\langle\boldsymbol{\xi}|\boldsymbol{\eta}\rangle_n = \int\frac{\mathrm{d}^2\boldsymbol{\xi}}{(2\pi)^n}|\boldsymbol{\xi}\rangle_n\exp\left[-\mathrm{i}\left(\boldsymbol{\eta}_1^T\boldsymbol{\xi}_2 - \boldsymbol{\eta}_2^T\boldsymbol{\xi}_1\right)\right], \tag{25}$$

where $\mathrm{d}^2\boldsymbol{\xi} = \mathrm{d}^2\xi_1\,\mathrm{d}^2\xi_2\cdots\mathrm{d}^2\xi_n$, $\boldsymbol{\eta}_1 = (\eta_{11}, \eta_{12}, \ldots, \eta_{1n})^T$, $\boldsymbol{\eta}_2 = (\eta_{21}, \eta_{22}, \ldots, \eta_{2n})^T$, $\boldsymbol{\xi}_1 = (\xi_{11}, \xi_{12}, \ldots, \xi_{1n})^T$, $\boldsymbol{\xi}_2 = (\xi_{21}, \xi_{22}, \ldots, \xi_{2n})^T$. From equation (25) we know that ${}_n\langle\boldsymbol{\xi}|\boldsymbol{\eta}\rangle_n$ is a Fourier transformation kernel, so $|\boldsymbol{\xi}\rangle_n$ is conjugate to $|\boldsymbol{\eta}\rangle_n$. Furthermore, $|\boldsymbol{\xi}\rangle_n$ satisfies the following eigenvalue equations:

$$(Q_A + Q_B)|\boldsymbol{\xi}\rangle_n = \sqrt{2}\boldsymbol{\xi}_1|\boldsymbol{\xi}\rangle_n, \tag{26}$$

and

$$(P_A - P_B)|\boldsymbol{\xi}\rangle_n = \sqrt{2}\boldsymbol{\xi}_2|\boldsymbol{\xi}\rangle_n. \tag{27}$$

The $2n$-mode squeezing operator $U(\mathbf{M})$ can be expressed in the representation $|\boldsymbol{\xi}\rangle_n$ as

$$\begin{aligned} U(\mathbf{M}) &= \sqrt{\det(\mathbf{M}^\dagger\mathbf{M})}\int\frac{\mathrm{d}^2\eta}{\pi^n}|\mathbf{M}\boldsymbol{\eta}\rangle_{nn}\langle\boldsymbol{\eta}| \\ &= \sqrt{\det(\mathbf{M}^\dagger\mathbf{M})}\int\frac{\mathrm{d}^2\boldsymbol{\eta}}{\pi^n}\int\frac{\mathrm{d}^2\boldsymbol{\xi}}{(2\pi)^n}\frac{\mathrm{d}^2\boldsymbol{\xi}'}{(2\pi)^n}|\boldsymbol{\xi}\rangle_{nn}\langle\boldsymbol{\xi}'| \\ &\quad\times\exp\left[-\mathrm{i}\left((\mathbf{M}\boldsymbol{\eta})_1^T\boldsymbol{\xi}_2 - (\mathbf{M}\boldsymbol{\eta})_2^T\boldsymbol{\xi}_1 - \boldsymbol{\eta}_1^T\boldsymbol{\xi}_2' + \boldsymbol{\eta}_2^T\boldsymbol{\xi}_1'\right)\right] \\ &= \sqrt{\det(\mathbf{M}^\dagger\mathbf{M})}\int\frac{\mathrm{d}^2\boldsymbol{\xi}}{(2\pi)^n}\frac{\mathrm{d}^2\boldsymbol{\xi}'}{(2\pi)^n}|\boldsymbol{\xi}\rangle_{nn}\langle\boldsymbol{\xi}'|\frac{(2\pi)^{2n}}{\pi^n}\delta(\boldsymbol{\xi}' - \mathbf{M}^\dagger\boldsymbol{\xi}) \\ &= \sqrt{\det(\mathbf{M}^\dagger\mathbf{M})}\int\frac{\mathrm{d}^2\boldsymbol{\xi}}{\pi^n}|\boldsymbol{\xi}\rangle_{nn}\left\langle\mathbf{M}^\dagger\boldsymbol{\xi}\right|. \end{aligned} \tag{28}$$

Hence the transformation on $Q_A + Q_B$ and $P_A - P_B$ induced by $U(\mathbf{M})$ is

$$\begin{aligned} U^\dagger(\mathbf{M})(Q_A + Q_B)U(\mathbf{M}) &= \sqrt{\det(\mathbf{M}^\dagger\mathbf{M})}U^\dagger(\mathbf{M})(Q_A + Q_B)\int\frac{\mathrm{d}^2\boldsymbol{\xi}}{\pi^n}|\boldsymbol{\xi}\rangle_{nn}\langle\mathbf{M}^\dagger\boldsymbol{\xi}| \\ &= \sqrt{\det(\mathbf{M}^\dagger\mathbf{M})}U^\dagger(\mathbf{M})\int\frac{\mathrm{d}^2\boldsymbol{\xi}}{\pi^n}\sqrt{2}\boldsymbol{\xi}_1|\boldsymbol{\xi}\rangle_{nn}\langle\mathbf{M}^\dagger\boldsymbol{\xi}| \\ &= \mathbf{N}_1^T(Q_A + Q_B) + \mathbf{N}_2^T(P_A - P_B) \end{aligned} \tag{29}$$

and

$$\begin{aligned} U^\dagger(\mathbf{M})(P_A - P_B)U(\mathbf{M}) &= \sqrt{\det(\mathbf{M}^\dagger\mathbf{M})}U^\dagger(\mathbf{M})(P_A - P_B)\int\frac{\mathrm{d}^2\boldsymbol{\xi}}{\pi^n}|\boldsymbol{\xi}\rangle_{nn}\langle\mathbf{M}^\dagger\boldsymbol{\xi}| \\ &= \sqrt{\det(\mathbf{M}^\dagger\mathbf{M})}U^\dagger(\mathbf{M})\int\frac{\mathrm{d}^2\boldsymbol{\xi}}{\pi^n}\sqrt{2}\boldsymbol{\xi}_2|\boldsymbol{\xi}\rangle_{nn}\langle\mathbf{M}^\dagger\boldsymbol{\xi}| \\ &= \mathbf{N}_1^T(P_A - P_B) - \mathbf{N}_2^T(Q_A + Q_B), \end{aligned} \tag{30}$$

where for the simplicity of writing we have introduced $\mathbf{N} = \mathbf{N}_1 + \mathrm{i}\mathbf{N}_2 \equiv \mathbf{M}^{-1}$.

Similar to (19), we can combine equations (29) and (30) to

$$U^{\dagger}(\mathbf{M})\mathbf{O}'U(\mathbf{M}) = \mathbf{N}^{\dagger}\mathbf{O}', \tag{31}$$

where

$$\mathbf{O}' = (Q_A + Q_B) + \mathrm{i}(P_A - P_B). \tag{32}$$

5. The compact form of $U(\mathbf{M})$ and its physical interpretation

In order to show the squeezing behavior of $U(\mathbf{M})$ more clearly, based on equations (17), (18), (29) and (30), we can write down the transformation on $\mathbf{A}^{\dagger}$, $\mathbf{A}$, $\mathbf{B}^{\dagger}$, $\mathbf{B}$:

$$U^{\dagger}(\mathbf{M})(\mathbf{A}^{\dagger T}, \mathbf{B}^{\dagger T}, \mathbf{A}^{T}, \mathbf{B}^{T})U(\mathbf{M}) = (\mathbf{A}^{\dagger T}, \mathbf{B}^{\dagger T}, \mathbf{A}^{T}, \mathbf{B}^{T})\begin{pmatrix} \frac{\mathbf{M}^{\dagger}+\mathbf{N}}{2} & 0 & 0 & \frac{\mathbf{N}-\mathbf{M}^{\dagger}}{2} \\ 0 & \frac{\mathbf{M}^{T}+\mathbf{N}^{*}}{2} & \frac{\mathbf{N}^{*}-\mathbf{M}^{T}}{2} & 0 \\ 0 & \frac{\mathbf{N}^{*}-\mathbf{M}^{T}}{2} & \frac{\mathbf{M}^{T}+\mathbf{N}^{*}}{2} & 0 \\ \frac{\mathbf{N}-\mathbf{M}^{\dagger}}{2} & 0 & 0 & \frac{\mathbf{M}^{\dagger}+\mathbf{N}}{2} \end{pmatrix}. \tag{33}$$

Thus according to [12], we obtain the compact form of $U(\mathbf{M})$ straightforwardly:

$$U(\mathbf{M}) = \exp\left[-\tfrac{1}{2}(\mathbf{A}^{\dagger T}, \mathbf{B}^{\dagger T}, \mathbf{A}^{T}, \mathbf{B}^{T})\mathbf{\Gamma}(\mathbf{A}^{\dagger T}, \mathbf{B}^{\dagger T}, \mathbf{A}^{T}, \mathbf{B}^{T})^{T}\right], \tag{34}$$

where

$$\begin{aligned}\mathbf{\Gamma} &= \ln\begin{pmatrix} \frac{\mathbf{M}^{\dagger}+\mathbf{N}}{2} & 0 & 0 & \frac{\mathbf{N}-\mathbf{M}^{\dagger}}{2} \\ 0 & \frac{\mathbf{M}^{T}+\mathbf{N}^{*}}{2} & \frac{\mathbf{N}^{*}-\mathbf{M}^{T}}{2} & 0 \\ 0 & \frac{\mathbf{N}^{*}-\mathbf{M}^{T}}{2} & \frac{\mathbf{M}^{T}+\mathbf{N}^{*}}{2} & 0 \\ \frac{\mathbf{N}-\mathbf{M}^{\dagger}}{2} & 0 & 0 & \frac{\mathbf{M}^{\dagger}+\mathbf{N}}{2} \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \\ &= \ln\gamma \cdot \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.\end{aligned} \tag{35}$$

We note that the matrix γ can be block diagonalized by the orthogonal matrix

$$\mathbf{R} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{36}$$

$$\gamma = \mathbf{R}\,\mathrm{diag}\{\mathbf{M}^{\dagger}, \mathbf{M}^{T}, \mathbf{N}^{*}, \mathbf{N}\}\mathbf{R}^{-1}. \tag{37}$$

So we can calculate the exact form of the matrix $\mathbf{\Gamma}$:

$$\mathbf{\Gamma} = \begin{pmatrix} 0 & \frac{\ln\mathbf{M}+\ln\mathbf{M}^{\dagger}}{2} & \frac{\ln\mathbf{M}^{\dagger}-\ln\mathbf{M}}{2} & 0 \\ \frac{\ln\mathbf{M}^{T}+\ln\mathbf{M}^{*}}{2} & 0 & 0 & \frac{\ln\mathbf{M}^{T}-\ln\mathbf{M}^{*}}{2} \\ \frac{\ln\mathbf{M}^{*}-\ln\mathbf{M}^{T}}{2} & 0 & 0 & \frac{-\ln\mathbf{M}^{*}-\ln\mathbf{M}^{T}}{2} \\ 0 & \frac{\ln\mathbf{M}-\ln\mathbf{M}^{\dagger}}{2} & \frac{-\ln\mathbf{M}-\ln\mathbf{M}^{\dagger}}{2} & 0 \end{pmatrix}. \tag{38}$$

Finally, we have the compact form of the operator

$$U(\mathbf{M}) = \exp\left[\mathbf{B}^T \frac{\ln \mathbf{M} + \ln \mathbf{M}^\dagger}{2}\mathbf{A} - \mathbf{A}^{\dagger T}\frac{\ln \mathbf{M} + \ln \mathbf{M}^\dagger}{2}\mathbf{B}^\dagger \right.$$
$$\left. + \mathbf{A}^{\dagger T}\frac{\ln \mathbf{M} - \ln \mathbf{M}^\dagger}{2}\mathbf{A} - \mathbf{B}^T \frac{\ln \mathbf{M} - \ln \mathbf{M}^\dagger}{2}\mathbf{B}^\dagger\right]. \tag{39}$$

For simplicity, we define $\mathbf{\Lambda} = \frac{\ln \mathbf{M}+\ln \mathbf{M}^\dagger}{2}$ as the Hermitian part of $\ln \mathbf{M}$ and $\mathrm{i}\mathbf{\Theta} = \frac{\ln \mathbf{M}-\ln \mathbf{M}^\dagger}{2}$ as the anti-Hermitian part of $\ln \mathbf{M}$, i.e.

$$\begin{aligned} &\mathbf{\Lambda}^\dagger = \mathbf{\Lambda}, \qquad \mathbf{\Theta}^\dagger = \mathbf{\Theta} \\ &\ln \mathbf{M} = \mathbf{\Lambda} + \mathrm{i}\mathbf{\Theta}; \end{aligned} \tag{40}$$

we have the more compact form of equation (39)

$$U(\mathbf{M}) = \exp[\mathbf{B}^T \mathbf{\Lambda} A - A^{\dagger T}\mathbf{\Lambda} B^\dagger + \mathrm{i}(\mathbf{A}^{\dagger T}\mathbf{\Theta} A - B^T \mathbf{\Theta} B^\dagger)]. \tag{41}$$

Since $\mathbf{\Lambda}$ and $\mathbf{\Theta}$ are Hermitian, we can find $n \times n$ unitary matrices u_1, u_2 that diagonalize $\mathbf{\Lambda}$ and $\mathbf{\Theta}$, respectively,

$$\begin{aligned} &u_1 \mathbf{\Lambda} u_1^\dagger = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\} \\ &u_2 \mathbf{\Theta} u_2^\dagger = \mathrm{diag}\{\theta_1, \ldots, \theta_n\}. \end{aligned} \tag{42}$$

If $\mathbf{\Theta} = 0$, that means $\mathbf{M} = \exp \Lambda$ is positive definite, then

$$\begin{aligned} U(\mathbf{M}) &= \exp[\mathbf{B}^T \mathbf{\Lambda} A - A^{\dagger T}\mathbf{\Lambda} B^\dagger] \\ &= \exp\left[-\sum_{j=1}^n \lambda_j \left(\mathbf{A}_j'^\dagger \mathbf{B}_j'^\dagger - \mathbf{A}_j' \mathbf{B}_j'\right)\right] \\ &= \prod_{j=1}^n \exp\left[-\lambda_j \left(\mathbf{A}_j'^\dagger \mathbf{B}_j'^\dagger - \mathbf{A}_j' \mathbf{B}_j'\right)\right], \end{aligned} \tag{43}$$

where

$$\begin{aligned} &\mathbf{A}' = u_1 \mathbf{A}, \mathbf{B}' = u_1^* \mathbf{B}, \\ &[\mathbf{A}_i', \mathbf{A}_j'] = [\mathbf{B}_i', \mathbf{B}_j'] = 0 \\ &\left[\mathbf{B}_i', \mathbf{A}_j^{\dagger\prime}\right] = \left[\mathbf{A}_i', \mathbf{B}_j^{\dagger\prime}\right] = 0 \\ &\left[\mathbf{A}_i', \mathbf{A}_j^{\dagger\prime}\right] = \left[\mathbf{B}_i', \mathbf{B}_j^{\dagger\prime}\right] = \delta_{ij}. \end{aligned} \tag{44}$$

Comparing equation (43) with equation (6), we can find that $U(\mathbf{M})$ describes pure squeezing between the new modes $\mathbf{A}'$ and $\mathbf{B}'$ with squeezing parameters $-\lambda_i$.

Similarly, if $\mathbf{\Lambda} = 0$, that means $\mathbf{M} = \exp[\mathrm{i}\mathbf{\Theta}]$ is unitary, then

$$\begin{aligned} U(\mathbf{M}) &= \exp[\mathrm{i}(\mathbf{A}^{\dagger T}\mathbf{\Theta} A - B^T \mathbf{\Theta} B^\dagger)] \\ &= \exp\left[\mathrm{i}\sum_{j=1}^n \theta_j \left(\mathbf{A}_j''^\dagger \mathbf{A}_j'' - \mathbf{B}_j''^\dagger \mathbf{B}_j''\right)\right] \\ &= \prod_{j=1}^n \exp\left[\mathrm{i}\theta_j \left(\mathbf{A}_j''^\dagger \mathbf{A}_j'' - \mathbf{B}_j''^\dagger \mathbf{B}_j''\right)\right], \end{aligned} \tag{45}$$

where

$$\begin{aligned}
&\mathbf{A}'' = u_2\mathbf{A}, \mathbf{B}'' = u_2^*\mathbf{B} \\
&[\mathbf{A}_i'', \mathbf{A}_j''] = [\mathbf{B}_i'', \mathbf{B}_j''] = 0 \\
&\left[\mathbf{B}_i'', \mathbf{A}_j^{\dagger\prime\prime}\right] = \left[\mathbf{A}_i'', \mathbf{B}_j^{\dagger\prime\prime}\right] = 0 \\
&\left[\mathbf{A}_i'', \mathbf{A}_j^{\dagger\prime\prime}\right] = \left[\mathbf{B}_i'', \mathbf{B}_j^{\dagger\prime\prime}\right] = \delta_{ij}.
\end{aligned} \tag{46}$$

Equation (45) shows that if $\mathbf{\Lambda} = 0$, then $U(\mathbf{M})$ describes the pure simultaneous phase shift with equal angles θ_j but the opposite direction of new modes $\mathbf{A}''$ and $\mathbf{B}''$. In general cases that $\mathbf{\Lambda} \neq 0$ and $\mathbf{\Theta} \neq 0$, $U(\mathbf{M})$ describes the mixed effect of both squeezing and phase shift.

Since it is $\mathbf{\Lambda}$ and $\mathbf{\Theta}$ that describe the physics in $U(\mathbf{M})$, not $\mathbf{M}$ itself, and the real $\mathbf{M}$ does not guarantee $\mathbf{\Theta} = 0$, it makes no sense to restrict $\mathbf{M}$ to be real. It is appropriate to define $U(\mathbf{M})$ for the arbitrary nonsingular complex matrix $\mathbf{M}$.

6. The squeezing properties of the 2*n*-mode squeezing operator

Theoretically, the $2n$-mode squeezed state is constructed by the $2n$-mode squeezing operator acting on the $2n$-mode vacuum state; it follows that

$$\begin{aligned}
U(\mathbf{M})|\mathbf{00}\rangle &= \frac{2^n\sqrt{\det(\mathbf{M}^\dagger\mathbf{M})}}{\det(\mathbf{1}+\mathbf{M}^\dagger\mathbf{M})} \exp[\mathbf{A}^{\dagger T}(1 - 2\mathbf{M}(\mathbf{1}+\mathbf{M}^\dagger\mathbf{M})^{-1}\mathbf{M}^\dagger)\mathbf{B}^\dagger]|\mathbf{00}\rangle \\
&\equiv \|\mathbf{00}\rangle.
\end{aligned} \tag{47}$$

The corresponding optical quadrature phase amplitudes can be expressed as follows:

$$X_1 = \frac{1}{2\sqrt{n}}\sum_{i=1}^{2n} Q_i, \qquad X_2 = \frac{1}{2\sqrt{n}}\sum_{i=1}^{2n} P_i, \qquad [X_1, X_2] = \frac{i}{2}. \tag{48}$$

The variances of the $2n$-mode quadrature are defined as

$$(\Delta X_i)^2 = \langle\mathbf{00}\|X_i^2\|\mathbf{00}\rangle - (\langle\mathbf{00}\|X_i\|\mathbf{00}\rangle)^2, \qquad i = 1, 2. \tag{49}$$

From equations (18) and (29), we have

$$\begin{aligned}
\langle\mathbf{00}\|X_1\|\mathbf{00}\rangle &= \langle\mathbf{00}|U^\dagger(\mathbf{M})X_1U(\mathbf{M})|\mathbf{00}\rangle \\
&= \frac{1}{2\sqrt{n}}\sum_{j=1}^{n}\langle\mathbf{00}|U^\dagger(\mathbf{M})\,(Q_A+Q_B)_j\,U(\mathbf{M})|\mathbf{00}\rangle \\
&= \frac{1}{2\sqrt{n}}\sum_{j,k=1}^{n}\langle\mathbf{00}|\left(\mathbf{N}_1^T\right)_{jk}(Q_A+Q_B)_k + \left(\mathbf{N}_2^T\right)_{jk}(P_A-P_B)_k|\mathbf{00}\rangle = 0
\end{aligned} \tag{50}$$

and

$$\begin{aligned}
\langle\mathbf{00}\|X_2\|\mathbf{00}\rangle &= \langle\mathbf{00}\|U^\dagger(\mathbf{M})X_2U(\mathbf{M})\|\mathbf{00}\rangle \\
&= \frac{1}{2\sqrt{n}}\sum_{j=1}^{n}\langle\mathbf{00}|U^\dagger(\mathbf{M})(P_A+P_B)_jU(\mathbf{M})|\mathbf{00}\rangle \\
&= \frac{1}{2\sqrt{n}}\sum_{j,k=1}^{n}\langle\mathbf{00}|\mathbf{M}_{1jk}(P_A+P_B)_k + \mathbf{M}_{2jk}(Q_A-Q_B)_k|\mathbf{00}\rangle = 0.
\end{aligned} \tag{51}$$

The variances of the quadrature in $\|\mathbf{00}\rangle$ are

$$\begin{aligned}
(\Delta X_1)^2 &= \langle\mathbf{00}\|X_1^2\|\mathbf{00}\rangle \\
&= \frac{1}{4n}\langle\mathbf{00}|U^\dagger(M)\left[\sum_{k=1}^{n}(Q_A+Q_B)_k\right]^2 U(M)|\mathbf{00}\rangle \\
&= \frac{1}{4n}\sum_{j,k,\alpha,\beta=1}^{n}\langle\mathbf{00}|\left[\left(\mathbf{N}_1^T\right)_{j\alpha}(Q_A+Q_B)_\alpha+\left(\mathbf{N}_2^T\right)_{j\alpha}(P_A-P_B)_\alpha\right] \\
&\quad\cdot\left[\left(\mathbf{N}_1^T\right)_{k\beta}(Q_A+Q_B)_\beta+\left(\mathbf{N}_2^T\right)_{k\beta}(P_A-P_B)_\beta\right]|\mathbf{00}\rangle \\
&= \frac{1}{4n}\sum_{j,k,\alpha,\beta=1}^{n}\left(\mathbf{N}_1^T\right)_{j\alpha}\left(\mathbf{N}_1^T\right)_{k\beta}\langle\mathbf{00}|(Q_A+Q_B)_\alpha(Q_A+Q_B)_\beta|\mathbf{00}\rangle \\
&\quad+\frac{1}{4n}\sum_{j,k,\alpha,\beta=1}^{n}\left(\mathbf{N}_2^T\right)_{j\alpha}\left(\mathbf{N}_2^T\right)_{k\beta}\langle\mathbf{00}|(P_A-P_B)_\alpha(P_A-P_B)_\beta|\mathbf{00}\rangle \\
&\quad+\frac{1}{4n}\sum_{j,k,\alpha,\beta=1}^{n}\left(\mathbf{N}_1^T\right)_{j\alpha}\left(\mathbf{N}_2^T\right)_{k\beta}\langle\mathbf{00}|(Q_A+Q_B)_\alpha(P_A-P_B)_\beta|\mathbf{00}\rangle \\
&\quad+\frac{1}{4n}\sum_{j,k,\alpha,\beta=1}^{n}\left(\mathbf{N}_2^T\right)_{j\alpha}\left(\mathbf{N}_1^T\right)_{k\beta}\langle\mathbf{00}|(P_A-P_B)_\alpha(Q_A+Q_B)_\beta|\mathbf{00}\rangle \\
&= \frac{1}{4n}\sum_{j,k,\alpha=1}^{n}\left[\left(\mathbf{N}_1^T\right)_{j\alpha}\left(\mathbf{N}_1^T\right)_{k\alpha}+\left(\mathbf{N}_2^T\right)_{j\alpha}\left(\mathbf{N}_2^T\right)_{k\alpha}\right] \\
&= \frac{1}{4n}\sum_{j,k=1}^{n}\left(\mathbf{N}_1^T\mathbf{N}_1+\mathbf{N}_2^T\mathbf{N}_2\right)_{jk} \\
&= \frac{1}{4n}\sum_{j,k=1}^{n}(\mathbf{N}^\dagger\mathbf{N})_{jk},
\end{aligned} \tag{52}$$

where we have used the identities

$$\begin{aligned}
&\langle\mathbf{00}|(Q_A+Q_B)_\alpha(Q_A+Q_B)_\beta|\mathbf{00}\rangle = \delta_{\alpha\beta}, \\
&\langle\mathbf{00}|(P_A-P_B)_\alpha(P_A-P_B)_\beta|\mathbf{00}\rangle = \delta_{\alpha\beta}, \\
&\langle\mathbf{00}|(Q_A+Q_B)_\alpha(P_A-P_B)_\beta|\mathbf{00}\rangle = 0, \\
&\langle\mathbf{00}|(P_A-P_B)_\alpha(Q_A+Q_B)_\beta|\mathbf{00}\rangle = 0.
\end{aligned} \tag{53}$$

Similarly,

$$\begin{aligned}
(\Delta X_2)^2 &= \langle\mathbf{00}\|X_2^2\|\mathbf{00}\rangle = \frac{1}{4n}\langle\mathbf{00}\|\left[\sum_{k=1}^{n}(P_A+P_B)_k\right]^2\|\mathbf{00}\rangle \\
&= \frac{1}{4n}\sum_{j,k,,\alpha,\beta=1}^{n}\langle\mathbf{00}|[\mathbf{M}_{1j\alpha}(P_A+P_B)_\alpha+\mathbf{M}_{2j\alpha}(Q_A-Q_B)_\alpha] \\
&\quad\times[\mathbf{M}_{1k\beta}(P_A+P_B)_\beta+\mathbf{M}_{2k\beta}(Q_A-Q_B)_\beta]|\mathbf{00}\rangle \\
&= \frac{1}{4n}\sum_{j,k,\alpha,\beta=1}^{n}\mathbf{M}_{1j\alpha}\mathbf{M}_{1k\beta}\langle\mathbf{00}|(P_A+P_B)_\alpha(P_A+P_B)_\beta|\mathbf{00}\rangle
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4n}\sum_{j,k,\alpha,\beta=1}^{n}\mathbf{M}_{2j\alpha}\mathbf{M}_{2k\beta}\langle\mathbf{00}|(Q_A-Q_B)_\alpha(Q_A-Q_B)_\beta|\mathbf{00}\rangle\\
&+\frac{1}{4n}\sum_{j,k,\alpha,\beta=1}^{n}\mathbf{M}_{1j\alpha}\mathbf{M}_{2k\beta}\langle\mathbf{00}|(P_A+P_B)_\alpha(Q_A-Q_B)_\beta|\mathbf{00}\rangle\\
&+\frac{1}{4n}\sum_{j,k,\alpha,\beta=1}^{n}\mathbf{M}_{2j\alpha}\mathbf{M}_{1k\beta}\langle\mathbf{00}|(Q_A-Q_B)_\alpha(P_A+P_B)_\beta|\mathbf{00}\rangle\\
&=\frac{1}{4n}\sum_{j,k,\alpha=1}^{n}[\mathbf{M}_{1j\alpha}\mathbf{M}_{1k\alpha}+\mathbf{M}_{2j\alpha}\mathbf{M}_{2k\alpha}]\\
&=\frac{1}{4n}\sum_{j,k=1}^{n}\left(\mathbf{M}_1\mathbf{M}_1^T+\mathbf{M}_2\mathbf{M}_2^T\right)_{jk}\\
&=\frac{1}{4n}\sum_{j,k=1}^{n}(\mathbf{M}\mathbf{M}^\dagger)_{jk},
\end{aligned}
\tag{54}
$$

where we have used the identities

$$
\begin{aligned}
&\langle\mathbf{00}|(P_A+P_B)_\alpha(P_A+P_B)_\beta|\mathbf{00}\rangle=\delta_{\alpha\beta},\\
&\langle\mathbf{00}|(Q_A-Q_B)_\alpha(Q_A-Q_B)_\beta|\mathbf{00}\rangle=\delta_{\alpha\beta}\\
&\langle\mathbf{00}|(P_A+P_B)_\alpha(Q_A-Q_B)_\beta|\mathbf{00}\rangle=0,\\
&\langle\mathbf{00}|(Q_A-Q_B)_\alpha(P_A+P_B)_\beta|\mathbf{00}\rangle=0.
\end{aligned}
\tag{55}
$$

As a general conclusion in quantum mechanics, for the arbitrary state $|\psi\rangle$,

$$
\langle\psi|(\Delta X_1)^2|\psi\rangle\langle\psi|(\Delta X_2)^2|\psi\rangle\geqslant\tfrac{1}{4}\langle\psi||[X_1,X_2]|^2|\psi\rangle=\tfrac{1}{16},
\tag{56}
$$

we wish to prove this inequality and figure out whether and when expression (29) can take equal sign for the squeezed vacuum state $|\psi\rangle=\|\mathbf{00}\rangle$.

For an arbitrary positive-definite matrix $\mathbf{S}=u^\dagger\,\mathrm{diag}\{s_1,\ldots,s_n\}u$, where u is a unitary matrix and s_i are the eigenvalues of $\mathbf{S}$, we have

$$
\sum_{i,j=1}^{n}\mathbf{S}_{ij}=\sum_{i,j,k,l=1}^{n}u_{ki}^*s_k\delta_{kl}u_{lj}=\sum_{i,j,k=1}^{n}s_ku_{ki}^*u_{kj}=\sum_{k=1}^{n}s_kc_k,
\tag{57}
$$

where for simplicity we define n non-negative numbers

$$
c_k=\sum_{i,j=1}^{n}u_{ki}^*u_{kj}=\left|\sum_{i=1}^{n}u_{ki}\right|^2\geqslant 0.
\tag{58}
$$

c_i satisfy the identity

$$
\sum_{i=1}^{n}c_i=\sum_{i,j,k=1}^{n}u_{ki}^*u_{kj}=\sum_{i,j=1}^{n}\delta_{ij}=n.
\tag{59}
$$

If we define the row-vectors of the unitary matrix u as $u_{i\_}=\{u_{i1},\ldots,u_{in}\}$ and $v_0=\{1,\ldots,1\}$, equation (58) can be written as

$$
c_i=|u_{i\_}\cdot v_0|^2\,.
\tag{60}
$$

So

$$\left(\sum_{i,j=1}^{n}(\mathbf{S}^{-1})_{ij}\right)\left(\sum_{i,j=1}^{n}\mathbf{S}_{ij}\right) = \left(\sum_{i=1}^{n}\frac{1}{s_i}c_i\right)\left(\sum_{i=1}^{n}s_i c_i\right)$$
$$= \sum_{i=1}^{n}c_i^2 + \sum_{i>j}\left(\frac{s_i}{s_j}+\frac{s_j}{s_i}\right)c_i c_j$$
$$= \left(\sum_{i=1}^{n}c_i\right)^2 + \sum_{i>j}\left(\sqrt{\frac{s_i}{s_j}}-\sqrt{\frac{s_j}{s_i}}\right)^2 c_i c_j$$
$$= n^2 + \sum_{i>j}\left(\sqrt{\frac{s_i}{s_j}}-\sqrt{\frac{s_j}{s_i}}\right)^2 c_i c_j$$
$$\geqslant n^2. \tag{61}$$

Hence from (52) and (54) we have (note $\mathbf{N}\equiv\mathbf{M}^{-1}$)

$$(\Delta X_1)^2(\Delta X_2)^2 = \left[\frac{1}{4n}\sum_{j,k=1}^{n}(\mathbf{N}^\dagger\mathbf{N})_{jk}\right]\left[\frac{1}{4n}\sum_{j,k=1}^{n}(\mathbf{M}\mathbf{M}^\dagger)_{jk}\right]$$
$$= \left(\frac{1}{4n}\right)^2\left[\sum_{j,k=1}^{n}((\mathbf{M}\mathbf{M}^\dagger)^{-1})_{jk}\right]\left[\sum_{j,k=1}^{n}(\mathbf{M}\mathbf{M}^\dagger)_{jk}\right]$$
$$\geqslant \left(\frac{1}{4n}\right)^2 n^2 = \frac{1}{16}, \tag{62}$$

where in the last step we considered $\mathbf{M}\mathbf{M}^\dagger = \mathbf{S}$, a positive definite matrix, and used equation (61).

In order to take equal sign in (61), a trivial condition is that all the eigenvalues of $\mathbf{M}\mathbf{M}^\dagger$ are the same, $s_i \equiv s$, i.e. $\mathbf{M}\mathbf{M}^\dagger = s\mathbf{I}$, which is not very interesting. In the case that $s_i \neq s_j$ for all $i \neq j$, we must have $c_i c_j \equiv 0$. Since $\sum_{i=1}^{n} c_i = n$, there must be an i_0 s.t. $c_{i_0} \neq 0$, so $c_i \equiv 0$ for all $i \neq i_0$. By definition, $c_i = |u_{i\_} \cdot v_0|^2$, in other words, vectors $u_{i\_}$ are orthogonal to v_0 for $i \neq i_0$. Since u is a unitary matrix, the row vectors of u are orthogonal to each other and form a complete set of the n-dimensional complex vector space. This implies that the i_0th row of u is $u_{i_0-} = \frac{\mathrm{e}^{\mathrm{i}\varphi}}{\sqrt{n}}\{1,\ldots,1\}$, where $\mathrm{e}^{\mathrm{i}\varphi}$ is an irrelevant phase factor. Thus, in the case that there is one row $u_{i_0-} = \frac{\mathrm{e}^{\mathrm{i}\varphi}}{\sqrt{n}}\{1,\ldots,1\}$ in the matrix u, we have $(\Delta X_1)^2(\Delta X_2)^2 \equiv \frac{1}{16}$, particularly, the variances take the simple form $(\Delta X_1)^2 = 1\big/\big(4s_{i_0}\big)$, $(\Delta X_2)^2 = s_{i_0}/4$, which indicates that $\|\mathbf{00}\rangle$ is the correct $2n$-mode squeezed state.

6.1. Physical implications

Through the above discussions we can see clearly the relationship between the multimode squeezed state and the entangled state; thus, if one wants to investigate entanglement involved in a many-body system, one may give rise to the multimode squeezed state. On the other hand, due to the orthonormal property of $|\boldsymbol{\eta}\rangle_n$ as shown in equation (9), we have

$$U(\mathbf{M})|\boldsymbol{\eta}\rangle_n = \sqrt{\det(\mathbf{M}^\dagger\mathbf{M})}|\mathbf{M}\boldsymbol{\eta}\rangle_n,$$

so the squeezing operator squeezes the entangled state $|\boldsymbol{\eta}\rangle_n$ in a natural way. Thus, the wavefunction of the squeezed vacuum state in the entangled state representation can easily be derived:

$$ {}_n\langle\boldsymbol{\eta}\|\mathbf{00}\rangle = {}_n\langle\boldsymbol{\eta}|\sqrt{\det(\mathbf{M}^{\dagger}\mathbf{M})}\int\frac{\mathrm{d}^2\boldsymbol{\eta}'}{\pi^n}|\mathbf{M}\boldsymbol{\eta}'\rangle_{nn}\langle\boldsymbol{\eta}'|\mathbf{00}\rangle = {}_n\langle\mathbf{M}^{-1}\boldsymbol{\eta}|\mathbf{00}\rangle, $$

so the best representation for studying various non-classical properties of the multimode squeezing state is $|\boldsymbol{\eta}\rangle_n$.

7. Summary

In summary, we have developed the concept of the bipartite EPR entangled state to the case of $2n$ particle systems. With the help of the IWOP technique, we have constructed the $2n$-mode squeezing operator which squeezes the n-pair entangled state and the squeezing properties of it are also discussed for $\mathbf{M}$ being Hermitian. We have seen that there is an intrinsic relationship between quantum entanglement and quantum squeezing for the multimode case through constructing the entangled state representation $|\boldsymbol{\eta}\rangle_n$. Its further applications in the study of multipartite teleportation, quantum dense coding and entangled fractional Fourier transformation are under consideration. We expect that the multimode squeezing effect may be observed or implemented in multiphoton fluorescence, multiphoton ionization and two-photon lasing processes.

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